MATH 5061 Lecture on 3/11/2020

Recall: $\pi : E \to M^m$ vector bundle, one consider

"frame bundle" $GL(r) \circlearrowleft$

$\hat{\pi}^{-1}(U) \subseteq F(E) \ni (p, \underset{\approx}{s})$ where $\underset{\approx}{s} = (s_1, \ldots, s_r)$ basis for E_p

$$\begin{tikzcd} & \hat{\pi}^{-1}(U) \subseteq F(E) \arrow[dl, "\varphi_U"'] \arrow[d, "\hat{\pi}"] \arrow[dr, "\hat{\pi}"] & \\ U \times GL(r) & U \subseteq M^m \ni p & \end{tikzcd}$$

$GL(r) \circlearrowright U \times GL(r)$

This is an example of "Principal $GL(r)$-bundle":

- $\exists$ free right $GL(r)$-action on $F(E)$ by "change of basis":

$$A \in GL(r) \Rightarrow R_A : F(E) \to F(E)$$
$$R_A(p, \underset{\approx}{s}) = (p, \underset{\approx}{s} A)$$

More generally, consider principal G-bundle for any Lie group G

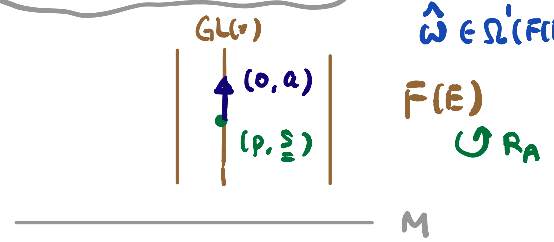
- locally, $\exists$ trivialization

$$\varphi_U : \hat{\pi}^{-1}(U) \xrightarrow{\cong} U \times GL(r)$$

s.t. it's "compatible" with $GL(r)$-action

$$\varphi_U(p, \underset{\approx}{s}) = (p, B)$$
$$\varphi_U(R_A(p, \underset{\approx}{s})) = (p, BA)$$

$\hat{\omega} \in \Omega^1(F(E)) \otimes gl(r)$



Thm: A connection D on $\pi : E \to M$ $\iff$ A connection 1-form $\hat{\omega}$ on $F(E)$ with values in $gl(r) = \{r \times r \text{ matrices}\}$ = Lie algebra of $GL(r)$

locally: $\omega = (\omega^i_j)$ 1-forms matrix-valued on M

s.t. (1) $\hat{\omega}_{(p,\underset{\approx}{s})}(0,a) = a \quad \forall (0,a) \in T_{(p,\underset{\approx}{s})} F(E)$

(2) $R_A^* \hat{\omega} = A^{-1} \hat{\omega} A \quad \forall A \in GL(r)$

Proof: "$\Leftarrow$" Exercise.

"$\Rightarrow$" Given D on $\pi : E \to M$, GOAL: Construct $\hat{\omega}$, satisfying (1) & (2).

$$\begin{array}{ccc} F(E) & & \hat{\pi}^*(\omega_U) \\ \downarrow \hat{\pi} & & \uparrow \hat{\pi}^* \\ M \supseteq U, & & \omega_U \end{array}$$

Locally, fix local basis $\underset{\approx}{s}_U = (s_1^U, \ldots, s_r^U)$ of E over $U \subseteq M$.

$\rightsquigarrow$ ω_U = connection 1-form, $gl(r)$-valued on U

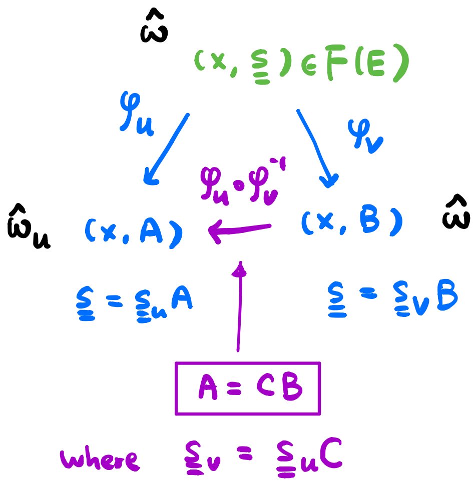
Idea: Define $\hat{\omega}_U := \hat{\pi}^*(\omega_U)$, 1-form, $gl(r)$-valued on $\hat{\pi}^{-1}(U)$

Q: well-defined? "invariance under coord. change"? ✓

Well-definedness: Suppose we have another local frame

$\underline{\underline{s}}_V := (s_1^V, \dots, s_r^V)$ of E over $V \subseteq M$ s.t. $U \cap V \neq \phi$

Claim: $\hat{\pi}^*(\omega_U) = \hat{\pi}^*(\omega_V)$ on $\hat{\pi}^{-1}(U \cap V) \subseteq F(E)$



Locally,

$$\hat{\omega}_U = \underbrace{A^{-1}dA}_{\text{tangent to fiber}} + \underbrace{A^{-1}\omega_U A}_{\text{tangent to } M} \qquad \text{in } \varphi_U\text{-coord.}$$

$$\hat{\omega}_V = B^{-1}dB + B^{-1}\omega_V B \qquad \text{in } \varphi_V\text{-coord.}$$

<u>Need to check:</u> $(\varphi_U \circ \varphi_V^{-1})^* \hat{\omega}_U = \hat{\omega}_V$

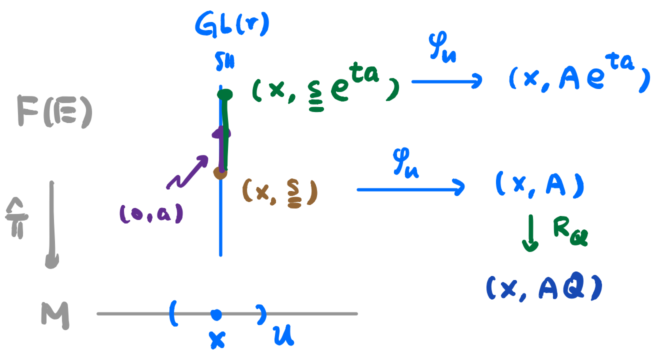
$$(\varphi_U \circ \varphi_V^{-1})^* \hat{\omega}_U \overset{A=CB}{=} (CB)^{-1}d(CB) + (CB)^{-1}\omega_U(CB)$$

$$= B^{-1}(C^{-1}dC)B + B^{-1}C^{-1}C\,dB + B^{-1}C^{-1}\omega_U CB$$

$$= B^{-1}dB + B^{-1}(\underbrace{C^{-1}dC + C^{-1}\omega_U C}_{\omega_V})B = \hat{\omega}_V.$$

Claim: $\hat{\omega}$ satisfies (1) & (2)

(1) $\boxed{\hat{\omega}(0, a) = a \quad \forall\, (0, a) \in T_{(x, \underline{\underline{s}})}F(E)}$ ✓



<u>Recall:</u> $\hat{\omega}_U = A^{-1}dA + A^{-1}\omega_U A$

At $(x, \underline{\underline{s}})$,

$$\hat{\omega}_{(x,\underline{\underline{s}})}(0, a) = \hat{\omega}_{U,(x,A)}(0, Aa) = A^{-1}(Aa) = a$$

(2) $\boxed{\forall\, Q \in GL(r), \quad R_Q^* \hat{\omega} = Q^{-1}\hat{\omega} Q.}$

<u>Recall:</u> $\hat{\omega}_U = A^{-1}dA + A^{-1}\omega_U A$ locally

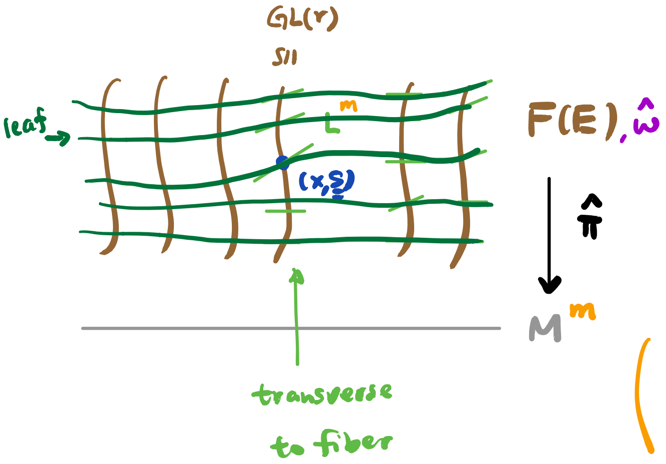
$$R_Q^*(\hat{\omega}_U) = (AQ)^{-1}d(AQ) + (AQ)^{-1}\omega_U(AQ)$$

$$= Q^{-1}(A^{-1}dA + A^{-1}\omega_U A)Q = Q^{-1}\hat{\omega}_U Q$$

$\uparrow$ Q is fixed

$\square$

This Thm. provides a different point of view to understand "connections".



connection $\hat{\omega}$ on $F(E)$ $\Longleftrightarrow$ connection D on E $\rightsquigarrow$ distribution L on $F(E)$

$$L^m := \{ (v,a) \in T_{(x,\underline{s})} F(E) \mid \hat{\omega}(v,a) = 0 \}$$

as $r \times r$ matrix

$\Big($ dim: $\dim L = m + r^2 - r^2 = m$

$\because \hat{\omega}(0,a) = a \Rightarrow$ non-deg. $\Big)$

Proposition: Locally, TFAE:

(1) D is flat (i.e. $\Omega \equiv 0$)

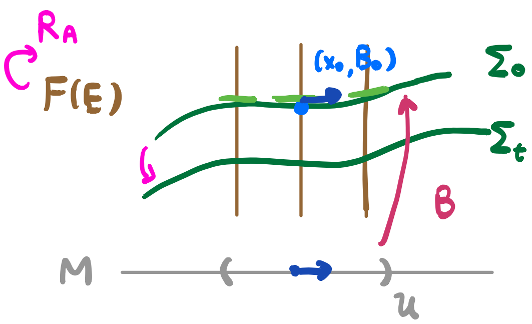
$\Leftrightarrow$ (2) L is integrable

$\Leftrightarrow$ (3) $\exists\, B : \mathcal{U} \to F(E)$ s.t. $B^*(\hat{\omega}) = 0$

i.e. $\exists$ parallel local frame $\tilde{\underline{s}} = (\tilde{s}_1, \ldots, \tilde{s}_r)$.

Proof by Pictures:

(2) $\Leftrightarrow$ (3):



$\Sigma_0 = \mathrm{graph}(B) = \{ (x, B(x)) \}$

integral submfd to L $\Rightarrow$ $\hat{\omega}\big|_{T\Sigma_0} \equiv 0$ $\Rightarrow$ $B^*\hat{\omega} = 0$

(1) $\Leftrightarrow$ (2): Recall: $\hat{\Omega} = d\hat{\omega} + \hat{\omega} \wedge \hat{\omega}$

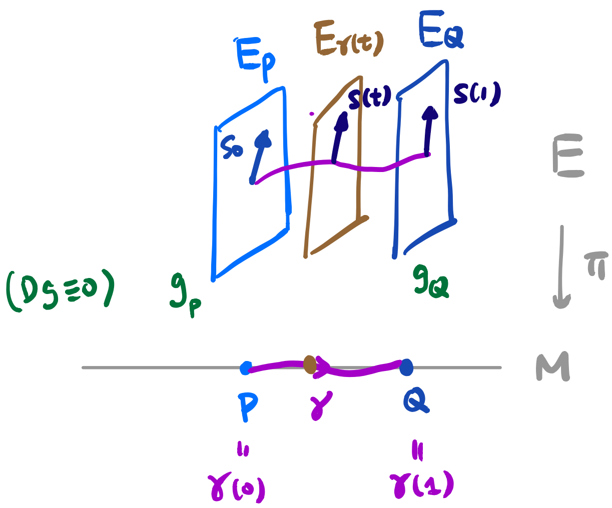
dual Frobenius thm.

$v, w \in L \Rightarrow d\hat{\omega}(v,w) = \underbrace{\hat{\Omega}(v,w) = 0}_{\text{flat}} \Leftrightarrow L$ integrable

$\square$

Parallel Transport

Given a connection D on $\pi : E \to M$. $\rightsquigarrow$ defines covariant derivative $D_X s$ on $s \in \Gamma(E)$



The parallel transport along γ :

$$P^{Q}_{P,\gamma} : E_P \xrightarrow{\cong} E_Q \qquad \text{linear isomorphism}$$
$$s_0 \longmapsto s(1)$$

$(Dg = 0 \Rightarrow \text{isometry})$

where $s : [0,1] \to E$ is the sol^{n} to

$$(\#) \begin{cases} D_{\dot\gamma} s = 0 \text{ along } \gamma \\ s(0) = s_0 \end{cases}$$

1st order ODE linear system

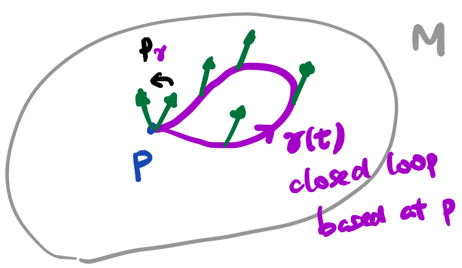
Locally: $(\#)$:

$$\begin{cases} \dfrac{d a^{\alpha}}{dt} + \Gamma^{\beta}_{i\alpha}(x(t)) \dfrac{dx^i}{dt} a^{\alpha} = 0 , & \alpha = 1,\dots,r \\ a^{\alpha}(0) = a^{\alpha}_0 . \end{cases}$$

unknown

where $\gamma'(t) = \sum_i \dfrac{dx^i}{dt} \dfrac{\partial}{\partial x^i}(\gamma(t))$; $s(t) = \sum_{\alpha=1}^{r} a^{\alpha}(t)\, s_{\alpha}(\gamma(t))$.

this leads to the notion of holonomy group :



$$P_\gamma : E_p \to E_p \in GL(E_p)$$

Define: $H_p(D) := \{ P_\gamma \mid \gamma \text{ loop at } p \} \subseteq GL(E_p)$

Some Remarks :

(i) $H_p \cong H_q \quad \forall p, q \in M$ (up to conjugate)

(ii) $Dg \equiv 0 \Rightarrow H_p \subseteq O(E_p)$

(iii) D flat $\underset{\text{locally}}{\Leftrightarrow}$ $H_p = \{1\}$

Affine Connections (Chern §4.2)

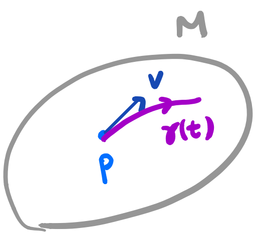
Def^{n}: A connection D on the tangent bundle TM is called an **affine connection**.

Note:
- general theory of connections on vector bundle applies.
- But often "more non-linear"

$$\begin{array}{ccc} E \ni (x^1, \dots, x^m, s_1, \dots, s_r) & \qquad & TM \ni (x^1, \dots, x^m, \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}) \\ \downarrow & & \downarrow \\ M \ni (x^1, \dots, x^m) & & M \ni (x^1, \dots, x^m) \end{array}$$

Geodesic eq^{n}: $\boxed{D_{\gamma'}\gamma' = 0}$ where γ is a parametrized curve on M (called geodesic)

In local coord, this is



$$\frac{d^2x^k}{dt^2} + \sum_{i,j=1}^{m} \Gamma_{ij}^k(x(t)) \frac{dx^i}{dt}\frac{dx^j}{dt} = 0$$

2nd order non-linear ODE system

(short-time existence & uniqueness)

Def^{n}: Given an affine connection D on TM, define its **torsion** to be

$$T : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

$$T(X,Y) := D_X Y - D_Y X - [X,Y]$$

Easy Facts:
- T is tensorial in both X and Y
- T is skew-symmetric (i.e. $T(Y,X) = -T(X,Y)$)

$$\Rightarrow \qquad T \in \Gamma(\Lambda^2 T^*M \otimes TM)$$

Alternatively, one can look at the

"tautological 1-form": $\theta \in \Gamma(T^*M \otimes TM)$

$\theta(X) := X$ i.e. $\theta = id_{TM}$

Prop: $\boxed{T = D\theta}$ where $D: \Gamma(\Lambda^1 T^*M \otimes TM) \to \Gamma(\Lambda^2 T^*M \otimes TM)$

Proof: Check locally, $e_1, \dots, e_m$ basis for TM, $\theta^1, \dots, \theta^m$ dual basis for T^*M.

$$\theta = \sum_{i=1}^m \theta^i \otimes e_i$$

$$\Rightarrow \quad D\theta = \sum_{i=1}^m \Big(d\theta^i \otimes e_i - \theta^i \wedge \underbrace{De_i}_{\sum_{j=1}^m \omega^j{}_i e_j} \Big) = \sum_{i=1}^m \underbrace{\Big(d\theta^i - \sum_{j=1}^m \theta^j \wedge \omega^i_j \Big)}_{\text{2-form}} \otimes e_i$$

Claim: $\boxed{T = d\theta + \omega \wedge \theta}$

Pf Claim: $(d\theta + \omega \wedge \theta)(X, Y)$

$$= \sum_{i=1}^m \Big[d\theta^i(X,Y) + \sum_j \omega^i_j \wedge \theta^j (X,Y) \Big] e_i$$

$$= \sum_{i=1}^m \Big[\underline{X(\theta^i(Y)) - Y(\theta^i(X)) - \theta^i([X,Y])} + \sum_j \big(\omega^i_j(X)\, \theta^j(Y) - \omega^i_j(Y)\, \theta^j(X) \big) \Big] e_i$$

inv. formula for $d\theta^i$

$$= \sum_{i=1}^m \Big[\theta^i(D_X Y) - \theta^i(D_Y X) - \theta^i([X,Y]) \Big] e_i$$

$$= \sum_{i=1}^m \Big[\theta^i(T(X,Y)) \Big] e_i = T(X,Y).$$

□

Def^n: D is **torsion-free** iff $T \equiv 0$ (i.e. $T^k_{ij} = T^k_{ji}$)

Def^n: D, D' are **affine equivalent** iff D, D' has the same parametrized geodesics

Note: Given an arbitrary affine connection D on TM,

it may not be torsion-free.

But, $\exists$ D' affine equivalent to D, and D' is torsion-free

Why: Ex: $D'_X Y := D_X Y - \frac{1}{2} T(X,Y)$ $\quad$ [i.e. $\Gamma'^k_{ij} = \frac{1}{2}(\Gamma^k_{ij} + \Gamma^k_{ji})$]

Prop: (Existence of normal coordinates at $p \in M$)

Let D be an affine connection on TM.

$T(p) = 0$ at some $p \in M$ $\iff$ $\exists$ local coord. $x^1, \dots, x^m$ of M, centered at p, s.t. $D_{\partial_i} \partial_j (p) = 0 \quad \forall\, i,j = 1, \dots, m$

"Sketch of Proof": $\Leftarrow$ trivial

"$\Rightarrow$" (Similar to the proof of general $E \to M$)

Ex: $\tilde{\Gamma}^k_{ij} = \sum_p \left(\frac{\partial \tilde{x}^k}{\partial x^p} \frac{\partial^2 x^p}{\partial \tilde{x}^i \partial \tilde{x}^j} \right) + \sum_{p,q,r} \left(\frac{\partial \tilde{x}^k}{\partial x^p} \frac{\partial x^q}{\partial \tilde{x}^j} \frac{\partial x^r}{\partial \tilde{x}^i} \right) \Gamma^p_{rq}$

define: $x^k = \tilde{x}^k - \frac{1}{2} \Gamma^k_{ij}(0)\, \tilde{x}^i \tilde{x}^j$

$\square$

Def$^{\underline{n}}$: (M^m, g) Riemannian manifold

is a C^∞ manifold M^m with a fiber metric g on TM
(positive definite)

Remark: g only non-degenerate $\rightsquigarrow$ pseudo-Riemannian

Fundamental Thm of Riem. Geometry

Given a Riem. mfd (M,g). $\exists!$ connection D on TM s.t.

① $Dg \equiv 0$ metric-compatible

② $T \equiv 0$ torsion-free

$\Big\}$ "Levi-Civita" / "Riemannian" connection